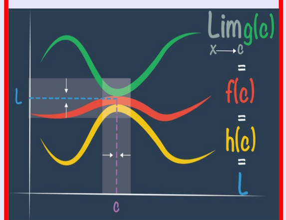
Math 261

Spring 2022

Lecture 10



Recall

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now $\quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

If we let $x = a+h$, then $h = x-a$

as $h \to 0$

$x - a \to 0$

$x \to a$

So $\quad \boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}}$

If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

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Since $f(x)$ is differentiable at $x=a$, this means that $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

Consider expression $f(x)-f(a)$

$$\frac{f(x)-f(a)}{x-a}\cdot(x-a)$$

Now let's take the lim as $x\to a$.

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right]$$

$$= \boxed{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}\cdot\lim_{x\to a}(x-a)$$

$$= f'(a)\cdot\lim_{x\to a}(x-a)$$

$$= f'(a)\cdot(a-a)$$

$$= f'(a)\cdot 0$$

$$= 0$$

$$\lim_{x\to a}[f(x)-f(a)]=0$$

$$\lim_{x\to a}f(x) - \lim_{x\to a}f(a)=0$$

$$\lim_{x\to a}f(x) - f(a) = 0$$

$$\boxed{\lim_{x\to a}f(x) = f(a)}$$

$\therefore$ $f(x)$ is cont. at $x=a$.

Find $f'(x)$

1) $f(x)=\frac{3}{4}x^8$

$f'(x)=\frac{3}{4}\cdot 8x^7$

$=\boxed{6x^7}$

2) $f(x)=\frac{1}{2}x^{-2}$

$f'(x)=\frac{1}{2}\cdot(-2x^{-3})=\boxed{-x^{-3}}$

3) $f(x)=\frac{\sqrt{x}}{x^2}=\frac{x^{\frac{1}{2}}}{x^2}=x^{\frac{1}{2}-2}=x^{\frac{-3}{2}}$

$\boxed{f'(x)=\frac{-3}{2}x^{-5/2}}$

4) $f(x)=\sqrt[3]{x}+4\sqrt{x^5}=x^{\frac{1}{3}}+4x^{\frac{5}{2}}$

$f'(x)=\frac{1}{3}x^{\frac{-2}{3}}+4\cdot\frac{5}{2}x^{\frac{3}{2}}$

$=\frac{1}{3}x^{-2/3}+10x^{\frac{3}{2}}$

Use product rule to find $f'(x)$

1) $f(x) = (\underbrace{2x^3+3})(\underbrace{x^4-2x})$

$f'(x) = 6x^2(x^4-2x) + (2x^3+3)\cdot(4x^3-2)$

$= 6x^6 - 12x^3 + 8x^6 - 4x^3 + 12x^3 - 6 = \boxed{}$

2) $f(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)(5x^3+x)$ $\to x \neq 0$

$f(x) = (\underbrace{x^{-2} - 3x^{-4}})(\underbrace{5x^3+x})$

$f'(x) = [-2x^{-3} + 12x^{-5}](5x^3+x) + (x^{-2}-3x^{-4})(15x^2+1)$

$= -10x^0 - 2x^{-2} + 60x^{-2} + 12x^{-4} + 15x^0 + x^{-2} - 45x^{-2} - 3x^{-4}$

$= -10 + 58x^{-2} + 9x^{-4} + 15 - 44x^{-2}$

$= 5 + 14x^{-2} + 9x^{-4} = \boxed{5 + \frac{14}{x^2} + \frac{9}{x^4}}$

Use the quotient rule to find $f'(x)$

1) $f(x) = \frac{x-3}{x+3}$

$f'(x) = \frac{1(x+3)-(x-3)\cdot 1}{(x+3)^2} = \boxed{\frac{6}{(x+3)^2}}$

2) $f(x) = \frac{x^3}{1-x^2}$ $\quad f'(x) = \frac{3x^2(1-x^2) - x^3(-2x)}{(1-x^2)^2}$

$= \frac{3x^2 - 3x^4 + 2x^4}{(1-x^2)^2} = \boxed{\frac{3x^2 - x^4}{(1-x^2)^2}}$

$f(x) = \frac{x}{x + \frac{c}{x}}$ $\quad f(x) = \frac{x^2}{x^2+c}$

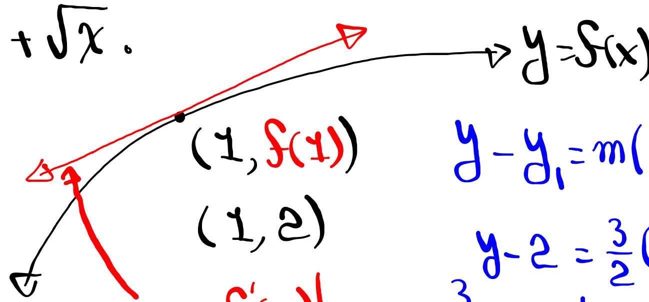
LCD $= x$

$f'(x) = \frac{2x(x^2+c) - x^2 \cdot 2x}{(x^2+c)^2}$

$\boxed{f'(x) = \frac{2xc}{(x^2+c)^2}}$

Find the equation of the tangent line at $x=1$

for $f(x) = x + \sqrt{x}$.



$f(x) = x + x^{\frac{1}{2}}$

$f'(x) = 1 + \frac{1}{2}x^{-\frac{1}{2}}$

$f'(1) = 1 + \frac{1}{2}(1)^{-1/2} = \frac{3}{2}$

$(1, f(1))$

$(1, 2)$

$m = f'(x)\Big|_{(1,2)} = \frac{3}{2}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{2}(x - 1)$

$\vdots$

$\boxed{y = \quad}$

Given $f(x) = \dfrac{1}{1+x^2}$

1) Find domain. $(-\infty, \infty)$

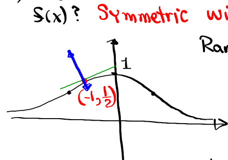
2) Find $\lim\limits_{x \to \infty} f(x) = 0$

3) Find $\lim\limits_{x \to -\infty} f(x) = 0$

4) Is $f(x)$ even, odd, or neither?
$f(-x) = f(x)$

5) How do we use the answer to (4) when drawing $f(x)$? Symmetric with respect to Y-axis.

Range $(0, 1]$



Find equation of the normal line to this curve at $x=-1$.

$m_{\text{Normal line}} = \dfrac{-1}{m_{\text{tan.line}}} = \dfrac{-1}{f'(x)\Big|_{(-1,\frac{1}{2})}}$

$f(x) = \dfrac{1}{1+x^2}$

$f'(x) = \dfrac{0(1+x^2) - 1 \cdot 2x}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$

$f'(-1) = \dfrac{2}{2^2} = \dfrac{1}{2}$

$m_{\text{Normal line}} = \dfrac{-1}{\frac{1}{2}} = \boxed{-2}$

Now $y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = -2(x - (-1)) \Rightarrow \boxed{y = \quad}$

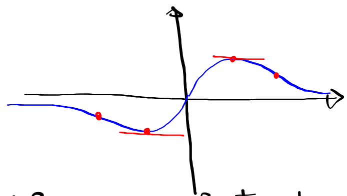
$f(x) = \dfrac{x}{x^2+1}$

1) Is $f(x)$ even, odd, or neither?

$f(-x) = \dfrac{-x}{(-x)^2+1} = \dfrac{-x}{x^2+1} = -\dfrac{x}{x^2+1} = -f(x)$

odd function
symmetric
w/t origin

2) Draw a rough graph of $f(x)$.



$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

3) Find eqn of tan. line at $x=3$

$m_{\text{tan.line}} = f'(x)\Big|_{(3,.3)}$

$(3, f(3)) = (3, .3)$
tan. Point

$f(x) = \dfrac{x}{x^2+1}$ $\quad f'(x) = \dfrac{1(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

$m = f'(3) = \dfrac{1-3^2}{(3^2+1)^2} = \dfrac{-8}{100} = \boxed{-.08}$

$y - y_1 = m(x - x_1)$ $\quad y - .3 = -.08(x-3) \rightarrow \boxed{y = \quad}$

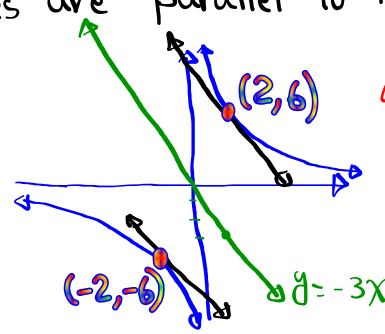
Find points on the graph of $xy = 12$ where tan. lines are parallel to the line $3x + y = 0$.

$\rightarrow y = \dfrac{12}{x}$

$y = -3x$

Y-Int $(0,0)$

$m = -3$



$y = 12x^{-1}$

$y' = -12x^{-2}$

Two lines are Parallel when have Same Slope

$f'(x) = -3$

$-12x^{-2} = -3$

$\dfrac{12}{x^2} = 3$

$x^2 = 4$

$x = \pm 2$

when $x = 2 \Rightarrow y = \dfrac{12}{2} = 6 \Rightarrow \boxed{(2,6)}$

when $x = -2 \Rightarrow y = \dfrac{12}{-2} = -6 \Rightarrow \boxed{(-2,-6)}$

Derivatives of Trig. Functions:

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\csc x] = -\csc x \cot x$

Show $\frac{d}{dx}[\csc x] = -\csc x \cot x$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{0 \cdot \sin x - 1 \cdot \cos x}{[\sin x]^2}$$

$$= \frac{-\cos x}{\sin x \cdot \sin x}$$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= \boxed{-\csc x \cdot \cot x}$$

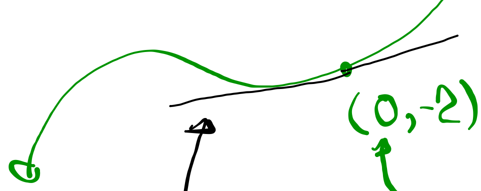
Show $\frac{d}{dx}\left[\cot x\right] = -\csc^2 x$

$$\frac{d}{dx}\left[\cot x\right] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$$

$$= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{[\sin x]^2} = \frac{-\sin^2 x - \cos^2 x}{[\sin x]^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}$$

Find equation of the tan. line to the graph of $f(x) = 3x^2 - 2\cos x$ at $x = 0$.



$f'(x) = 6x - 2\cdot(-\sin x)$

$= 6x + 2\sin x$

$m = f'(x)\big|_{(0,-2)}$

$m = f'(0) = 0$

$y - y_1 = m(x - x_1)$

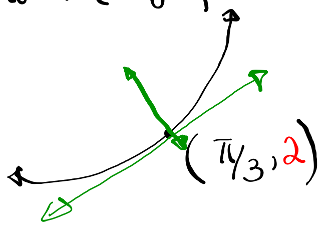
$y - (-2) = 0(x - 0) \Rightarrow \boxed{y = \quad}$

Find equation of the normal line at $x = \frac{\pi}{3}$ to the graph of $y = \sec x$.

60°

$y' = \sec x \tan x$

$\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{\frac{1}{2}} = 2$



$m = f'(x) \Big|_{(\frac{\pi}{3}, 2)} = \sec \frac{\pi}{3} \cdot \tan \frac{\pi}{3} = \sec 60^\circ \cdot \tan 60^\circ = 2 \cdot \sqrt{3} = 2\sqrt{3}$

tan. line

$m_{\text{Normal line}} = \frac{-1}{2\sqrt{3}} = \frac{-\sqrt{3}}{6}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{-\sqrt{3}}{6}\left(x - \frac{\pi}{3}\right)$

$\boxed{y = \quad}$

Evaluate

$$\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$$

Recall

$1 = \tan \frac{\pi}{4}$

$$f'(a) = \lim_{x \to \boxed{a}} \frac{\boxed{f(x)} - f(a)}{x - \boxed{a}}$$

$$\lim_{x \to \boxed{\frac{\pi}{4}}} \frac{\boxed{\tan x} - \tan \frac{\pi}{4}}{x - \boxed{\frac{\pi}{4}}}$$

$f(x) = \tan x$

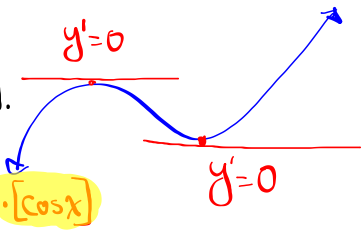
$f'(x) = \sec^2 x$

$$= f'\left(\frac{\pi}{4}\right) = \sec^2 \frac{\pi}{4} = \left[\sqrt{2}\right]^2 = \boxed{2}$$

March 14, 2022

Find all points on the graph of $y = \dfrac{\cos x}{2+\sin x}$ at which the tangent line is horizontal on $[0, 2\pi]$.

$m = 0$



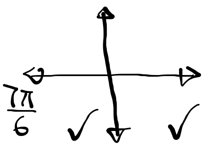
$$y' = \frac{-\sin x[2+\sin x] - \cos x \cdot [\cos x]}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - 1}{(2+\sin x)^2}$$

$y' = 0 \Rightarrow -2\sin x - 1 = 0$

$\sin x = \frac{-1}{2}$

Ref. Angle $30° = \frac{\pi}{6}$



QIII $\Rightarrow \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

QIV $\Rightarrow 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$\left(\frac{7\pi}{6}, ?\right)$

$\left(\frac{11\pi}{6}, ?\right)$

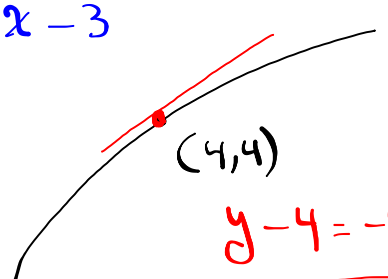
Class QZ 7

Find equation of the tan. line to the graph of $f(x) = \dfrac{4}{x-3}$ at $x = 4$.

$f(4) = \dfrac{4}{4-3} = \dfrac{4}{1} = \boxed{4}$

$f'(x) = \dfrac{0(x-3) - 4(1)}{(x-3)^2}$

$f'(x) = \dfrac{-4}{(x-3)^2}$ $\quad m = f'(4) = -4$



$y - 4 = -4(x-4)$

$\boxed{y = -4x + 20}$